Journal of Statistical Physics, Vol. 32, No. 2, 1983

Nonlinear Quantum Dynamical Semigroups for Many-Body Open Systems

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Received January 10, 1983; revised March 22, 1983

The notion of a nonlinear quantum dynamical semigroup is introduced, and the existence and uniqueness of solutions of the corresponding nonlinear evolution equations are studied in a more abstract framework. The construction of nonlinear quantum dynamical semigroups is carried out for two different mean-field models. First a mean-field coupling between a system of noninteracting subsystems and the bath is investigated. As examples, a nonlinear frictional Schrödinger equation and a model for a quantum Boltzmann equation are discussed. Second, a many-body system with mean-field interaction coupled to a bath is considered. Here, again, the form of the generator is derived; however, it cannot be obtained rigorously, except for some particular examples. Finally, the quantum Ising–Weiss model is briefly studied.

KEY WORDS: Quantum dynamical semigroups; open systems; reduced description of many-body systems; mean-field models; nonlinear evolution equations; nonlinear frictional Schrödinger equations; quantum Boltzmann equation; Hartree equation; Ising–Weiss model.

1. INTRODUCTION

In classical statistical mechanics the dynamics of most systems is governed by nonlinear kinetic equations on the phase space. Nonlinear evolution equations become exact in certain limiting situations from the underlying Hamiltonian dynamics. The Boltzmann equation, the Vlasov equation, and

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0022-4715/83/0800-0299$03.00/0

the Landau equation are rigorous consequences of the microscopic dynamics in the low-density, the mean-field, and the weak-coupling limits, respectively.[1] On an abstract level the structure was uncovered and studied by McKean[2,3] under the notion of nonlinear Markov processes.

In quantum theory of many-body systems one expects nonlinear quantum evolution equations. Known examples are the Hartree equation and its endowment with a linear dissipative generator for the Dicke–Haken–Lax laser model.[1]

Here we investigate a certain type of nonlinear evolution equations. Global existence, uniqueness, and positivity of the solutions are studied in section 2 in a general setup. In sections 3 and 4 the canonical forms of the nonlinear generators are derived and discussed for classes of open quantum mean-field models. Generally, such models as the laser or the Curie–Weiss–Ising or BCS model are described by classical differential equations for the expectation values of the observables in the Heisenberg picture.[4–6] In this sense we present a treatment in the Schrödinger picture which attempts to be a quantum generalization of McKean's nonlinear Markov processes.

2. NONLINEAR EVOLUTION EQUATIONS

For classes of quantum many-body systems we obtain in the next sections by mean-field and Markovian limits the following form of a nonlinear quantum evolution equation:

$$\dot{\rho} = L_0\rho + L[\rho]\rho \tag{1}$$

Here L_0 is a linear, in general unbounded generator of a quantum dynamical semigroup.[7,8] For given ρ, $L[\rho]$ is a bounded perturbation. Let $\mathscr{T}_+(\mathscr{H})$ be the Banach cone of positive trace-class operators on a separable Hilbert space $\mathscr{H}$. We define as *nonlinear quantum dynamical semigroup* a one-parameter family of maps Φ_t onto $\mathscr{T}_+(\mathscr{H})$ which fulfills the following conditions for $t \geqslant 0$:

(a) $\operatorname{tr}\Phi_t(\rho) = \operatorname{tr}\rho$

(b) $\Phi_s \circ \Phi_t = \Phi_{s+t}$

(c) For every $\rho \in \mathscr{T}_+(\mathscr{H})$ the function $t \to \Phi_t(\rho)$ is continuous.

In the following we investigate existence and uniqueness of global solutions of abstract evolution equations on Banach spaces. We aim at nonlinear quantum dynamical semigroups but the results are sufficiently general to include some classical dynamical systems.

Let X be an ordered Banach space and X_+ its positive cone. Let L_0 be the generator of a contracting positive semigroup with domain $\mathscr{D}(L_0)$ and $\mathscr{CP}(X) \subset \mathscr{B}(X)$ the convex set of bounded generators of contracting positive semigroups. We consider functions L on X with $L(x) \in \mathscr{B}(X)$ for every $x \in X$, such that $L(X_+) \subset \mathscr{CP}(X)$. We shall consider functions L subject to the following conditions:

(L1) $$\|L(x)\| \leqslant C_1(\|x\|)$$

(L2) $$\|L(x) - L(y)\| \leqslant C_2(\|x\|, \|y\|)\|x - y\|$$

for every $x, y \in X$. For functions L with $L(x)\mathscr{D}(L_0) \subset \mathscr{D}(L_0)$ for every $x \in \mathscr{D}(L_0)$ we use the following conditions:

($L_0$1) $$\|L_0 L(x)x\| \leqslant C_3(\|x\|)\|L_0 x\|$$

($L_0$2) $$\|L_0 L(x)x - L_0 L(y)y\| \leqslant C_4(\|x\|, \|L_0 x\|, \|y\|, \|L_0 y\|) \times \|L_0 x - L_0 y\|$$

where C_i $(i = 1, 2, 3, 4)$ are monotone increasing, everywhere finite functions. We consider the following differential equation:

$$\dot{x}_t = L_0 x_t + L(x_t)x_t, \qquad x_0 = x, \quad t \geqslant 0 \tag{2}$$

and its corresponding integral equation

$$x_t = e^{L_0 t}x + \int_0^t e^{L_0(t-s)}L(x_s)x_s\, ds \tag{3}$$

Let $T > 0$. For given $n \in \mathbb{N}$, $0 < \Delta \leqslant T$, we define a function $\Phi_t^{(n,\Delta)}$ on X by the recursive relation

$$\Phi_t^{(n,\Delta)}x = x \qquad \text{for} \quad t = 0 \tag{4}$$

and

$$\Phi_t^{(n,\Delta)}x = \exp\left\{\left(L_0 + L\left(\Phi_{t_k}^{(n,\Delta)}x\right)\right)(t - t_k)\right\}\Phi_{t_k}^{(n,\Delta)}x \qquad \text{for} \quad t \in \left[t_k, t_{k+1}\right] \tag{5}$$

where $\{t_k / k = 0, \ldots, n+1,\ t_0 = 0,\ t_{n+1} = T\}$ is a partition of the interval $[0, T]$ such that $|t_k - t_{k+1}| \leqslant \Delta$.

Theorem 1. (i) Under the conditions (L1) and (L2) there exists a unique global solution of the integral equation (3) for $x \in X_+$. Moreover, this solution x_t is positive, $\|x_t\| \leqslant \|x\|$, and

$$x_t = \underset{\substack{n \to \infty \\ \Delta \to 0}}{\text{s-lim}}\, \Phi_t^{(n,\Delta)}x$$

uniformly on compact intervals $[0, T]$ for arbitrary $T > 0$. (ii) If in addition the conditions ($L_0$1) and ($L_0$2) hold then the solution x_t to any $x \in \mathscr{D}(L_0) \cap X_+$ is continuously differentiable and fulfills the differential equation (2).

Proof. We use the standard theorems from fixed point theory (see, e.g., Ref. 9).

(i) It remains to show that

$$\underset{\substack{n\to\infty \\ \Delta\to 0}}{\text{s-lim}}\, \Phi_t^{(n,\Delta)} x = x_t$$

where x_t is the local solution of (3) for some $[0, T_1]$. Let

$$(Sx)_t = e^{L_0 t}x + \int_0^t e^{L_0(t-s)} L(x_s) x_s \, ds \tag{6}$$

$$(S^{(n,\Delta)}x)_t = e^{L_0 t}x + \int_0^t e^{L_0(t-s)} L(\hat{x}_s^{(n,\Delta)}) x_s \, ds \tag{7}$$

with $\hat{x}_s^{(n,\Delta)} = x_{t_k}$ for $s \in [t_k, t_{k+1}) \subset [0, T_1]$ where $\{t_k / k = 0, \ldots, n+1\}$ is a partition of $[0, T_1]$ with $|t_k - t_{k+1}| \leqslant \Delta$. Let T_1 be sufficiently small, and let $\mathscr{Y}(T_1) = \{x_{(\cdot)} : [0, T_1] \to X$, continuous and $\|x_{(\cdot)}\|_{T_1} = \sup_{t\in[0,T_1)} \|x_t\| < \infty\}$; then by (L1) and (L2) S and $S^{(n,\Delta)}$ are contractions on $\mathscr{Y}(T_1, \epsilon, x) \subset \mathscr{Y}(T_1)$, which consists of those $x_{(\cdot)}$ with $x_0 = x$ and $\|x_t - e^{L_0 t}x\| \leqslant \epsilon$ for $t \in [0, T_1)$. By inspection $\tilde{x}_t^{(n,\Delta)} = \Phi_t^{(n,\Delta)} x$ is a fixed point of $S^{(n,\Delta)}$ on $\mathscr{Y}(T_1, \epsilon, x)$ and furthermore for any $x_{(\cdot)} \in \mathscr{Y}(T_1, \epsilon, x)$

$$\lim_{\substack{n\to\infty \\ \Delta\to 0}} S^{(n,\Delta)} x_{(\cdot)} = S x_{(\cdot)} \tag{8}$$

Therefore by Ref. 10, Theorem V.21, $\tilde{x}_t^{(n,\Delta)}$ converges uniformly to x_t. By the approximation $\Phi_t^{(n,\Delta)}$ positivity is preserved and $\|x_t\| \leqslant \|x\|$ if $x \in X_+$. Consequently a global unique solution exists by Ref. 9, Theorem X.74.

(ii) The smoothness of the solution with $x \in \mathscr{D}(L_0)$ follows from Ref. 9, Theorem X.74. ■

Remark. There are existence theorems under weaker conditions than Lipschitz continuity. For example if the Banach space is finite dimensional and (L1) holds, one can construct a global, but not necessarily unique, solution by Schauder's fixed point theorem if $L(\cdot)$ is continuous on X only.

Example. Let L_m be a bounded dual generator of a completely positive semigroup[7] on $\bigotimes_{i=1}^m \mathscr{T}(\mathscr{H})$, $\mathscr{H}$ being a separable Hilbert space. Then it is immediate that the mapping $L : \mathscr{T}(\mathscr{H}) \ni \rho \to L(\rho) \in \mathscr{B}(\mathscr{T}(\mathscr{H}))$ defined by $L(\rho)\sigma = \mathrm{tr}_{[2,\ldots,m]}\{L_m \sigma \otimes \rho \otimes \cdots \otimes \rho\}$ maps $\mathscr{T}_+(\mathscr{H})$ into $\mathscr{CP}(\mathscr{T}(\mathscr{H}))$ and fulfills the properties (L1) and (L2). If L_0 is a dual generator of a completely positive semigroup, then the solution of the integral equation (3) with the $L(\rho)$ as above defines a nonlinear quantum dynamical semigroup.

3. MEAN-FIELD AND MARKOVIAN LIMITS FOR QUANTUM OPEN SYSTEMS I

We consider local observable algebras $\mathscr{A}_\Lambda = \bigotimes_{i\in\Lambda} \mathscr{A}_i$, where $\mathscr{A}_i = \mathscr{B}(\mathscr{H})$ is the C^*-algebra of bounded operators on a separable Hilbert space $\mathcal{H}$, and Λ is a finite subset of $\mathbb{N}$. $\mathcal{A}_\Lambda$ may be viewed as a subalgebra of $\bigotimes_{i\in\Lambda'}\mathcal{A}_i$ by the isotonic embedding $\mathcal{A}_i = \mathbb{1}$, if $i \in \Lambda'\backslash\Lambda$, for all finite $\Lambda' \supset \Lambda$. The quasilocal algebra associated to the net of local rings $\{\mathcal{A}_\Lambda/\Lambda \subset \mathbb{N}\}$ is denoted by $\mathcal{A}_\infty$. A state on $\mathcal{A}_\infty$ is *locally normal* if for every finite $\Lambda \subset \mathbb{N}$ the restriction ω_Λ is given by a density matrix. It is called *symmetric* if $\omega(A)$ does not depend on the order of the $X_i \in \mathcal{B}(\mathcal{H})$ for $A = X_1 \otimes \cdots \otimes X_n$, $n \in \mathbb{N}$. A state ω on $\mathcal{A}_\infty$ is a *product state* if for all finite $\Lambda_1 \subset \mathbb{N}$, $\Lambda_2 \subset \mathbb{N}$ with $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $X_i \in \mathcal{A}_{\Lambda_i}$ $(i = 1,2)$, $\omega(X_1X_2) = \omega(X_1)$ $\omega(X_2)$. Every symmetric locally normal state on $\mathcal{A}_\infty$ can be decomposed into normal product states

$$\omega = \int \mu(\omega \mid d\rho)\omega_\rho \tag{9}$$

with a unique measure $\mu(\cdot \mid d\rho)$ on $\mathcal{T}_{+,1}(\mathcal{H})$, the set of density matrices ρ on $\mathcal{H}$.[11–14]

In the mean-field limit the particles move independently in an average "medium" depending on the states of the particles. Let $\Lambda_N(t)$ denote the N-particle, in general irreversible, dynamics on $\mathcal{A}_\Lambda$ with $|\Lambda| = N$; then $\Lambda_N(t)$ has the *mean-field property* if

$$\operatorname*{s-lim}_{N\to\infty} \mathrm{tr}_{[m,N]}\Lambda_N(t)\rho\otimes \cdots \otimes \rho = \rho(t) \otimes \cdots \otimes \rho(t) \tag{10}$$

for every $\rho \in \mathcal{T}_+(\mathcal{H})$ and $t > 0$. $\mathrm{tr}_{[m,N]}$ denotes the partial trace over the Hilbert spaces labeled by $m, m+1, \ldots, N$, and tr_n denotes the partial trace over the nth Hilbert space. Here

$$\rho(t) = \Gamma_t\rho$$

is called *single-particle evolution*. Γ_t extends to a dynamics on the locally normal symmetric states by (9).

In the following we study models with mean-field dynamics. They are all covered by a general theorem.

Theorem 2. Let $\Lambda_N(t)$ be the dual of a completely positive quantum dynamical semigroup with generator $L_N^{(N)}$, where

$$L_n^{(N)} = -i\left[\sum_{i=1}^{n} h_i, \cdot\right] + \frac{1}{2N}\sum_{i,j=1}^{n} L_{ij} \tag{11}$$

h_i are identical copies of a self-adjoint single-particle Hamiltonian h. L_{ij} is a bounded operator on $\mathcal{T}(\mathcal{H}_i \otimes \mathcal{H}_j)$ for $i \neq j$. If $i < j$ (respectively, $i > j$), then the L_{ij} are identical copies of L_{12} (respectively, L_{21}). If $i = j$, then the L_{ij} are identical copies of the bounded operator L_{11} on $\mathcal{T}(\mathcal{H})$.

Then $\Lambda_N(t)$ has the mean-field property and the single-particle dynamics Γ_t is the solution of the following integral equation:

$$\rho(t) = S_1(t)\rho + \int_0^t S_1(t-s)\tfrac{1}{2}\mathrm{tr}_2\big[(L_{12} + L_{21})\rho(s)\otimes\rho(s)\big]\,ds \tag{12}$$

where $S_1(t)$ is defined by (17). If the solution of (12) is differentiable, then it satisfies the differential equation

$$\dot{\rho} = -i[h, \rho] + \tfrac{1}{2}\mathrm{tr}_2[(L_{12} + L_{21})\rho \otimes \rho] \tag{13}$$

Proof. Our theorem generalizes Theorem 5.7 of Ref. 1. Let $\rho_n^{(N)}(t) = \mathrm{tr}_{[n+1,N]}(\Lambda_N(t)\rho \otimes \cdots \otimes \rho)$; then

$$\dot{\rho}_n^{(N)} = L_n^{(N)}\rho_n^{(N)}(t) + \frac{N-n}{2N}\sum_{j=1}^{n} \mathrm{tr}_{n+1}(L_{jn+1} + L_{n+1j})\rho_{n+1}^{(N)}(t) \tag{14}$$

With the collision operator

$$C_{n,n+1} = \sum_{j=1}^{n} \mathrm{tr}_{n+1}(L_{jn+1} + L_{n+1j}) \tag{15}$$

the time-dependent perturbation series writes as (5.78) in Ref. 1, however, with

$$S_n^{(N)}(t) = \exp L_n^{(N)}t \tag{16}$$

Since $S_n^{(N)}(t)$ is a contraction semigroup the mth term of the perturbation series is bounded in trace norm $\|\cdot\|_n$ on $\mathscr{T}(\bigotimes_{j=1}^n \mathscr{H})$ by (5.79) of Ref. 1, if one replaces $4\|V\|$ by $\frac{1}{2}\|L_{12} + L_{21}\|$. Now, as shown in Ref. 1, the perturbation series converges in trace norm for all $t < t_0$ to the form (5.80) in Ref. 1, where

$$S_n(t)\rho_n = \operatorname*{s-lim}_{N\to\infty} S_n^{(N)}(t)\rho_n = e^{-iH_n t}\rho_n e^{iH_n t} \tag{17}$$

with $H_n = \sum_{i=1}^n h_i$. Then, as in Ref. 1, $\rho_n^{(N)}(t)$ converges for all t.

Since the conditions (L1) and (L2) are satisfied the integral equation (12) has a unique global solution $\rho(t)$. Using manifestly the construction of this solution, presented in the proof of Theorem 1, one finds that $\rho_n(t) = \bigotimes^n \rho(t)$ fulfills the hierarchy integral equation

$$\rho_n(t) = S_n(t)\rho_n + \int_0^t S_n(t-s)C_{n,n+1}\rho_{n+1}(s)\,ds \tag{18}$$

if $\rho_n = \bigotimes^n \rho$. By similar arguments as those mentioned above, (18) possesses a unique solution which fulfills (5.80) of Ref. 1. ■

In the following we discuss two examples of mean-field open systems. In both examples we consider a system with Hilbert space $\mathscr{H}_N = \bigotimes_{i=1}^N \mathscr{H}$ and Hamiltonian $\sum_{i=1}^N h_i$, with identical copies of h, coupled to a reservoir with Hilbert space $\mathscr{H}_\omega$, being the GNS representation space to a stationary state ω. The interaction λW_N is assumed to be a bounded, Hermitian operator. Following standard procedures,[15,8,1] we obtain after performing the weak or singular coupling limit a N-particle completely positive quantum dynamical semigroup with the dual generator $L_N^{(N)}$ of (11).

Example 1: Nonlinear Frictional Schrödinger Equation. We assume an interaction of the form

$$W_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} A_i \otimes \varphi \tag{19}$$

where the A_i's are identical copies of a bounded, Hermitian single-particle operator A, and φ is a bounded, Hermitian operator on $\mathscr{H}_\omega$. We think of (19) as a crude model of a dissipative process as it might occur in nuclear physics or quantum optics. A mean collective variable is coupled to a field coordinate of the bath. In both the weak and singular coupling limit we obtain the N-particle master equation of Lindblad type:[7]

$$\begin{aligned}\dot{\rho}_N = &-i\left[\sum_{j=1}^{N} h_i, \rho_N\right] - i\frac{1}{2N}\sum_{k\neq l=1}^{N}\left[B_{kl}, \rho_N\right] \\ &+ \frac{1}{2N}\sum_{\alpha}\sum_{k,l=1}^{N}\left\{\left[A_k^{(\alpha)}, \rho_N A_l^{(\alpha)*}\right] + \left[A_k^{(\alpha)}\rho_N, A_l^{(\alpha)*}\right]\right\}\end{aligned} \tag{20}$$

Here the B_{kl}'s are identical copies of a bounded Hermitian two-particle operator $B_{12} = B_{21}$. The manifest form of the $A^{(\alpha)}$'s and the B_{12} in terms of A, h, and the correlation function of the bath can be found in the literature.[15,16,8,1] By Theorem 2 and assuming differentiability of the solution we are led to the following single-particle dynamics:

$$\dot{\rho} = -i[h + H_1[\rho] + H_2[\rho], \rho] \tag{21}$$

$$H_1[\rho] = \mathrm{tr}_2 B_{12}\rho \tag{22}$$

$$H_2[\rho] = \frac{1}{2} i \sum_{\alpha}\left[\mathrm{tr}(A^{(\alpha)*}\rho)A^{(\alpha)} - \mathrm{tr}(A^{(\alpha)}\rho)A^{(\alpha)*}\right] \tag{23}$$

For a pure initial state the time evolution conserves purity. This gives a foundation to nonlinear frictional Schrödinger equations which have been introduced on phenomenological grounds.[17–19] If formally $A^{(\alpha)} = a^{(\alpha)}P + b^{(\alpha)}Q$, with $\sum_\alpha \mathrm{Im}\, b^{(\alpha)*}a^{(\alpha)} = \frac{1}{2}\gamma$, $B_{12} = 0$, $h = (1/2m)P^2 + V(Q) + \frac{1}{2}\gamma\{P, Q\}$ is inserted in (21)–(23) one obtains Süssmann's equation for friction constant γ and $c = 1/2$.[17–19]

Example 2: Stochastic Quantum Boltzmann Equation. Let us consider a N-particle system with Hamiltonian

$$H_N = \sum_{i=1}^{N} h_i \tag{24}$$

where $h_j = h$ are identical copies with $\exp(-\beta h)$ being trace-class for all $\beta > 0$. Let $\mathscr{H}_\omega = \bigotimes_{k>l=1}^{N} \mathscr{F}_\Omega$ be the state space of a bath where $\mathscr{F}_\Omega$ is a Hilbert space with distinguished vector Ω, e.g., the Fock space with the vacuum state. The coupling between the system and the bath is given by the interaction Hamiltonian

$$W_N = \frac{1}{\sqrt{N}} \sum_{k>l=1}^{N} Q_{kl} \otimes \varphi_{kl} \tag{25}$$

Here $Q_{kl} = Q_{21} \in \mathscr{B}(\mathscr{H} \otimes \mathscr{H})$ for all $l < k$, and $\varphi_{kl} = \varphi_{21} \in \mathscr{B}(\mathscr{F}_\Omega)$. Q_{21} is a symmetric operator with respect to permutations of particles. The Fourier-transformed two-point function of the bath is denoted by

$$\hat{h}(\epsilon) = \int_{-\infty}^{\infty} e^{i\epsilon t}(\Omega, \varphi_{21}\varphi_{21}(t)\Omega)\, dt \tag{26}$$

We furthermore assume that the level spacing for the single-particle Hamiltonian h is bounded from below by some positive number E_0, i.e., $|E - E'| \geqslant E_0 > 0$ for all different eigenvalues E, E', and that $\hat{h}(\epsilon) = 0$ for all $|\epsilon| \geqslant E_0$. Our ansatz is a mean-field version of Davies' model of heat conduction.[20] The physical meaning of our ansatz consists in replacing a direct interaction by potentials between the particles by the impact of some intermediating reservoirs. The above conditions prevent an exchange of energy between the particles and the baths in the weak-coupling limit and the influence of the reservoirs are of stochastic type.

After performing the weak-coupling limit[20] we obtain the following Markovian evolution:

$$\frac{d\rho_N}{dt} = -i\left[\sum_{j=1}^{N} h_j, \rho_N\right] + \frac{1}{2N} \sum_{k \neq l=1}^{N} L_{kl}\rho_N \tag{27}$$

with

$$L_{kl} = -i[B_{kl}, \cdot\,] + K_{kl} \tag{28}$$

and

$$K_{kl} = -[A_{kl}, [A_{kl}, \cdot\,]] \tag{29}$$

Here, again $A_{kl} = A_{12} = A_{21} \in \mathscr{B}(\mathscr{H} \otimes \mathscr{H})$, $B_{kl} = B_{12} = B_{21} \in \mathscr{B}(\mathscr{H} \otimes \mathscr{H})$ are Hermitian operators commuting with $h_1 + h_2$,

$$[A_{12}, h_1 \otimes \mathbb{1} + \mathbb{1} \otimes h_2] = [B_{12}, h_1 \otimes \mathbb{1} + \mathbb{1} \otimes h_2] = 0 \tag{30}$$

in the sense of commutations with the spectral projections. The manifest form of A_{12} and B_{12} is given in Ref. 20. Since (30), ($L_0$1), and ($L_0$2) are satisfied $\rho(t)$ is differentiable for every $\rho \in \mathscr{D}([h, \cdot])$ and according to

Theorem 2, the reduced single-particle dynamics is governed by the nonlinear stochastic quantum Boltzmann equation:

$$\dot{\rho} = -i[h + H_1[\rho], \rho] + \mathrm{tr}_2 K_{12}(\rho \otimes \rho) \tag{31}$$

with

$$H_1[\rho] = \mathrm{tr}_2 B_{12}\rho \tag{32}$$

The name "quantum Boltzmann equation" is justified by the following properties of the evolution equation (31), (32), which we prove for finite-dimensional Hilbert space $\mathscr{H}$:

(i) The "energy" $E = \mathrm{tr}(\rho(t)h)$ is a constant of motion. This follows from

$$\dot{E} = \mathrm{tr}\,\dot{\rho}h = \tfrac{1}{2}\mathrm{tr}_{12}\{L_{12}(\rho \otimes \rho)(h_1 \otimes \mathbb{1} + \mathbb{1} \otimes h_2)\} = 0 \tag{33}$$

using (30).

(ii) The H-theorem is valid:

$$\begin{aligned} -\frac{d}{dt}\left[\mathrm{tr}\,\rho(t)\ln\rho(t)\right] &= -\mathrm{tr}(\dot{\rho}\ln\rho) \\ &= -\tfrac{1}{2}\mathrm{tr}_{12}\left[\ln(\rho\otimes\rho)L_{12}(\rho\otimes\rho)\right] \geqslant 0 \end{aligned} \tag{34}$$

using Theorem 3 of Ref. 21 and Lemma 1 of Ref. 22.

(iii) The Gibbs states $\rho_\beta = Z(\beta)^{-1}\exp(-\beta h)$ are stationary states of (31) for $\beta > 0$.

These three properties carry over to the case of an infinite-dimensional Hilbert space $\mathscr{H}$, provided the time derivative of energy (33) and of entropy (34) exist.

4. MEAN-FIELD AND MARKOVIAN LIMITS FOR QUANTUM OPEN SYSTEMS II

In this part we study a model of interacting particles as the system, which is coupled to a bath. Now the mean-field scaling appears in the interaction of the particles of the system and not in the coupling. We perform the mean-field limit first and investigate afterwards the Markovian approximation of the resulting reduced Hartree equation. Here our arguments are not entirely rigorous.

The Hilbert space of the bath has the following structure:

$$\mathscr{H}_\omega = \bigotimes^{N} \mathscr{H}_\beta$$

where $\mathscr{H}_\beta$ is the GNS Hilbert space related to the KMS state ω_β at inverse temperature $\beta > 0$. The Hamiltonian in this representation is denoted by

H^B. The Hamiltonian of the total system is

$$H_N^\lambda = \sum_{j=1}^{N} \{h_j + H_j^B + \lambda Q_j \otimes \varphi_j\} + \frac{1}{2N} \sum_{k \neq l = 1}^{N} V_{kl} \tag{35}$$

Here, again, we use identical copies $h_j = h$, $H_j^B = H^B$, $Q_j = Q \in \mathscr{B}(\mathscr{H})$, $\varphi_j = \varphi \in \mathscr{B}(H_\beta)$, and $V_{kl} = V_{lk} = V_{12} \in \mathscr{B}(\mathscr{H} \otimes \mathscr{H})$. We perform first the limit $N \to \infty$. Using Theorem 2 one obtains

$$\lim_{N\to\infty} \mathrm{tr}_{[n+1,N]}\left(e^{-iH_N^\lambda t} \bigotimes^{N} \sigma e^{iH_N^\lambda t}\right) = \bigotimes^{n} \sigma(t) \tag{36}$$

where $\sigma(t)$ fulfills the integral equation associated with the following formal Hartree equation:

$$\frac{d\sigma(t)}{dt} = -i\left[h + H^B + H_1\left[\mathrm{tr}_R \sigma(t)\right] + \lambda Q \otimes \varphi, \sigma(t)\right] \tag{37}$$

where $H_1[\rho] = \mathrm{tr}_2 V_{12}\rho$, and tr_R is a partial trace over the Hilbert space $\mathscr{H}_\beta$. Next we eliminate the reservoirs. Let $\rho(t) = \mathrm{tr}_R \sigma(t)$ and $\sigma(0) = \rho(0) \otimes \omega_\beta$; then $\rho(t)$ may be written as

$$\rho(t) = \mathrm{tr}_R\left(T\exp\left\{-i\int_0^t H_1^\lambda[\rho(\tau)]\,d\tau\right\}\rho(0) \otimes \omega_\beta T\exp\left\{i\int_0^t H_1^\lambda[\rho(\tau)]\,d\tau\right\}\right) \tag{38}$$

where $H_1^\lambda[\rho] = h + H^B + H_1[\rho] + \lambda Q \otimes \varphi$. The above expression can be formally treated as a density matrix for an open system under the influence of the time-dependent potential $H_1^\lambda(\tau) \equiv H_1^\lambda[\rho(\tau)]$. Therefore one can apply the methods used in Ref. 23 to obtain the integral equation for $\rho(t)$, being "nonhomogeneous in time." We introduce, as usual, the projection technique:

$$\begin{aligned} Z[\rho]\rho &= -i[h + H_1[\rho], \rho] \\ A\sigma &= -i[Q \otimes \varphi, \sigma] \\ P_0\sigma = (\mathrm{tr}_R\sigma) \otimes \omega_\beta, &\qquad P_1 = \mathbb{1} - P_0 \\ A_{ij} = P_i A P_j, &\qquad i, j = 0, 1 \end{aligned} \tag{39}$$

Let

$$Z(t) \equiv Z[\rho(t)] \tag{40}$$

and

$$U_\lambda(t,s) = T\exp\int_s^t [Z(t') + \lambda A_{11}]\,dt' \tag{41}$$

The final result can be written as follows:

$$\rho(t) = U_0(t,0)\rho(0) + \lambda^2 \int_0^t ds \int_0^s du \left[U_0(t,s) A_{01} U_\lambda(s,u) A_{10} \rho(u) \right] \quad (42)$$

One attempt to obtain the Markovian approximation for (42) might use the usual weak-coupling limit method.[15,23] Unfortunately, because the variation of $Z(t)$ might be more rapid than the duration of the relaxation time of the open system, we are not able to prove in general the existence of the Markovian limit. However, a formal derivation from Eq. (42) leads to the following nonlinear master equation:

$$\frac{d\rho}{dt} = -i[h + H_1[\rho], \rho] + \lambda^2 L[\rho]\rho \quad (43)$$

where

$$L[\rho]\kappa = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} ds \int_0^\infty dt \Big\{ h^*(t) \Big[\big(\Phi[\rho]_{s-t}^{-1} Q\big), \kappa \big(\Phi[\rho]_s^{-1} Q\big) \Big] + h(t) \Big[\big(\Phi[\rho]_s^{-1} Q\big)\kappa, \big(\Phi[\rho]_{s-t}^{-1} Q\big) \Big] \Big\} \quad (44)$$

$$h(t) = \omega_\beta(\varphi\varphi(t)) \quad (45)$$

$$\frac{d}{dt}(\Phi[\rho]_t X) = -i[h + H_1[T_t\rho], (\Phi[\rho]_t X)], \qquad \Phi[\rho]_0 = \mathbb{1} \quad (46)$$

$$\frac{d}{dt} T_t\rho = -i[h + H_1[T_t\rho], T_t\rho], \qquad T_0 = \mathbb{1} \quad (47)$$

The same result follows from applying first the weak-coupling limit, which gives the generator K_N for a fixed N, and then the mean-field limit for $\mathrm{tr}_{[n+1,N]}\{K_N \rho \otimes \cdots \otimes \rho\}$. Although a precise treatment of the Markovian approximation is inaccessible, we are able to extract some rigorous information about the stationary states for our model in the weak-coupling limit.

Let $\tilde{\rho}_\lambda(t)$ be given by (38) with $\rho(0) = \rho_\infty$. The state ρ_∞ is called *stationary state in the weak-coupling limit* if

$$\lim_{\lambda\to 0} \left\{ \sup_{t\in[0,T]} \frac{1}{\lambda^2} \|\tilde{\rho}_\lambda(t) - \rho_\infty\|_1 \right\} = 0 \quad (48)$$

for all $T > 0$.

Theorem 3. Let $\dim \mathscr{H} < \infty$. The states which fulfill the following equation:

$$\rho_\infty = e^{-\beta(h + H_1[\rho_\infty])} / \mathrm{tr}\, e^{-\beta(h + H_1[\rho_\infty])} \quad (49)$$

are the stationary states in the weak-coupling limit. They are the only such states under the following conditions:

$$\text{(i)} \qquad \hat{h}(\epsilon) = \int_{-\infty}^{\infty} h(t) e^{i\epsilon t}\, dt > 0 \tag{50}$$

$$\text{(ii)} \qquad \{h + H_1[\rho], Q\}' = \{\mathbb{C}\mathbb{1}\} \qquad \text{for all} \quad \rho \in \mathscr{T}_{+,1}(\mathscr{H}) \tag{51}$$

Here $\{A\}'$ denotes the commutant of $\{A\}$.

Proof. Let ρ_∞ be a stationary state in the weak-coupling limit. Then one can expand the right-hand side of (42) with respect to λ:

$$\tilde{\rho}_\lambda(t) = \tilde{U}_0(t,0)\rho_\infty + \lambda^2 \int_0^t ds \int_0^s du \{\tilde{U}_0(t,s) A_{01} \tilde{U}_0(s,u) A_{10} \tilde{U}_0(u,0)\rho_\infty\} + O(\lambda^3) \tag{52}$$

where

$$\tilde{U}_0(t,s) = \exp\{-i[h + H_1[\rho_\infty], \cdot\,](t-s)\} \tag{53}$$

Taking into account (48) we have

$$[h + H_1[\rho_\infty], \rho_\infty] = 0 \tag{54}$$

Then we follow the usual approach[15] for time-independent Hamiltonians $H_\infty = h + H_1[\rho_\infty]$. One concludes that ρ_∞ is a stationary state for the linear generator K_∞,

$$K_\infty \rho = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} ds \int_0^\infty dt \{h^*(t)[Q_{s-t}^\infty, \rho Q_s^\infty] + h(t)[Q_s^\infty \rho, Q_{s-t}^\infty]\} \tag{55}$$

with

$$Q_t^\infty = e^{iH_\infty t} Q e^{-iH_\infty t}$$

The propositions of Theorem 3 are now consequences of the well-known results on stationary states of a quantum open system coupled to a heat bath.[24,25] ■

Remarks. Theorems 3 presents a dynamical approach to the problem of equilibrium states for mean-field models (compare with Ref. 26). In general the solutions of (49) are not unique. This is related to the occurrence of phase transitions in mean-field models. Among other conditions on the potential uniqueness of the solutions holds for temperatures high enough. The generator $K_\infty = L[\rho_\infty]$ [cf. (44)–(47)] describes the linearized evolution in the neighborhood of the stationary state ρ_∞. It may serve to investigate the stability of the stationary solutions.

Example: Ising–Weiss Model. We consider as a simple example the open Ising–Weiss model studied from a different point of view in Ref.

5. For this model the nonlinear generator $L[\rho]$ has a simple form and describes a well-defined nonlinear quantum dynamical semigroup. Let $\mathscr{H} = \mathbb{C}^2$, $h = 0$, $V_{12} = -J\sigma_1^3\sigma_2^3$, $Q = \sigma^1$. Let $\sigma_j^1, \sigma_j^2, \sigma_j^3$ be the Pauli matrices for the jth spin, and $\sigma^{\pm} = \frac{1}{2}(\sigma^1 \pm i\sigma^2)$. In this case

$$\begin{aligned} L[\rho]\kappa = \hat{h}\left(-2J\,\mathrm{tr}(\rho\sigma^3)\right)\left\{\left[\sigma^+, \kappa\sigma^-\right] + \left[\sigma^+\kappa, \sigma^-\right]\right\} \\ + \hat{h}\left(2J\,\mathrm{tr}(\rho\sigma^3)\right)\left\{\left[\sigma^-, \kappa\sigma^+\right] + \left[\sigma^-\kappa, \sigma^+\right]\right\} \\ + i\left\{s\left(2J\,\mathrm{tr}(\rho\sigma^3)\right) - s\left(-2J\,\mathrm{tr}(\rho\sigma^3)\right)\right\}\left[\sigma^3, \kappa\right] \end{aligned} \tag{56}$$

where

$$\int_0^\infty dt\, e^{i\epsilon t} h(t) = \frac{1}{2}\hat{h}(\epsilon) + is(\epsilon) \tag{57}$$

and

$$\hat{h}(-\epsilon) = \hat{h}(\epsilon)e^{-\beta\epsilon} \tag{58}$$

is the KMS condition for the bath. If $\hat{h}(\epsilon)$ and $s(\epsilon)$ are Lipschitz-continuous on $[-2J, 2J]$, then by Theorem 1 the equation of motion

$$\dot{\rho} = iJ\left[\mathrm{tr}(\rho\sigma^3)\sigma^3, \rho\right] + \lambda^2 L[\rho]\rho \tag{59}$$

possesses a unique global solution for all initial density matrices ρ_0. The stationary states for (59) are given by Eq. (49). For $J > 0$, $\hat{h}(\epsilon) > 0$, and $\beta > \beta_c = J^{-1}$ there are two stable stationary states and one unstable. For $\beta < \beta_c$ the stationary states are unique and stable (compare with Ref. 5).

ACKNOWLEDGMENT

One of the authors (R.A.) is grateful to the Alexander von Humboldt foundation for a research fellowship.

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